Propositional Satisfiability

This exercise sheet is mostly taken from the course notes of Pascale Le Gall and Marc Aiguier at CentraleSupélec.

Exercise 1 (Conversion to Normal Form)**.**

A formula is said to be in *negation normal form* if the negation operator $\neg$ appears only directly attached to a propositional variable. Literals include variables (i.e., p) and negations of variables (i.e., $\neg p$).
For example, $(p \vee \neg q) \wedge (\neg r \vee p)$ is in negation normal form, while $\neg(p \vee \neg q)$ is not.

1. Define a process to transform propositional formulas into equivalent formulas in negation normal form.

A formula is said to be in *conjunctive normal form* if it is expressed as a conjunction of one or more clauses, where a clause is defined as a disjunction of literals. For example, $(p \vee \neg q) \wedge (\neg r \vee p)$ is in conjunctive normal form, while $(p \wedge \neg q) \vee (\neg r \vee p)$ is not.

2. Convert the formula
$$\neg(((p \Rightarrow q) \Rightarrow \neg q) \Rightarrow \neg q)$$
into conjunctive normal form.

Exercise 2 (DPLL Algorithm)**.**

Starting from a formula in conjunctive normal form, the DPLL algorithm consists of successively:

1. Searching for unit clauses (reduced to a literal, i.e., of the form p or $\neg p$) and assigning a truth value to the propositional variable involved that makes the clause true, i.e., 1 (resp. 0) if the clause is reduced to p (resp. $\neg p$);
2. Searching for propositional variables that consistently appear in the same form (either as p or as $\neg p$): if they appear as p (resp. $\neg p$), they are said to have positive (resp. negative) polarity. If a variable has positive (resp. negative) polarity, the truth value 1 (resp. 0) is assigned to it;
3. Randomly choosing a variable p and successively assigning it both values (alternatively).

Steps 1, 2, and 3 are systematically attempted in this order: step $i + 1$ is only applied if step i cannot (or can no longer) be applied. If the DPLL algorithm returns the value 1 (resp. 0), the input formula is satisfiable (resp. unsatisfiable).
Apply the DPLL algorithm to each of the formulas below to determine whether they are satisfiable.

1. $(a \vee b \vee c \vee \neg d) \wedge (\neg a \vee b \vee \neg c) \wedge (\neg a \vee \neg b \vee c) \wedge (\neg a \vee \neg b \vee \neg c) \wedge d$

2.

$$
\begin{aligned}
&(a \lor \neg b \lor c \lor \neg d \lor \neg f \lor \neg h)\\
\land&(b \lor c)\\
\land&(b \lor c \lor d \lor e \lor h)\\
\land&(b \lor \neg d \lor e \lor \neg f)\\
\land&(b \lor \neg c \lor f \lor \neg h)\\
\land&(b \lor \neg d \lor f \lor \neg g)\\
\land&(\neg b \lor \neg c)\\
\land&(\neg b \lor \neg c \lor \neg d)\\
\land&(\neg b \lor c \lor d)\\
\land&(\neg b \lor \neg d \lor e)\\
\land&(\neg b \lor \neg e \lor f)\\
\land&(\neg g \lor \neg h)\\
\land&(g \lor h)\\
\land&(\neg g \lor h)
\end{aligned}
$$

Exercise 3 (Tseitin Transformation)**.**

1. For a, b, c propositional variables, provide a conjunctive normal form for $a \Leftrightarrow (b \land c)$, $a \Leftrightarrow (b \lor c)$, $a \Leftrightarrow \neg b$, $a \Leftrightarrow (b \Rightarrow c)$, where $\Leftrightarrow$ is the Boolean connector defined by

$$a \Leftrightarrow b \stackrel{def}{=} ((a \Rightarrow b) \land (b \Rightarrow a))$$

For any formula ϕ of one of the forms above, let $\mathsf{cnf}(\phi)$ denote the conjunctive normal form obtained. The function $\mathsf{Tseitin}$ is defined as

$$\mathsf{Tseitin}(\phi) = x_\phi \land T(\phi)$$

where $x_p = p$ and x_ϕ is a new variable if ϕ is not a variable, and where the function T is defined inductively by

- $T(p) = \top$
- $T(\neg\psi) = T(\psi) \land \mathsf{cnf}(x_\phi \Leftrightarrow \neg x_\psi)$
- $T(\psi @ \theta) = T(\psi) \land T(\theta) \land \mathsf{cnf}(x_\phi \Leftrightarrow x_\psi @ x_\theta)$ pour tout $@ \in \{\land, \lor, \Rightarrow\}$

2. Show that $\mathsf{Tseitin}(\phi)$ is a formula in conjunctive normal form.

3. The *size* of a formula ϕ, denoted $|\phi|$, is its number of connectives. Show that there exists a constant $K \in \mathbb{N}$ such that for any formula ϕ,

$$|\mathsf{Tseitin}(\phi)| \leq K|\phi| + 1$$

4. Show that ϕ and $\mathsf{Tseitin}(\phi)$ are equisatisfiable, i.e., one is satisfiable if and only if the other is satisfiable.

5. Provide the Tseitin transformation for the formula $\neg((a \lor b) \Rightarrow (b \land a))$.