



A Note on Łoś's Theorem for Kripke-Joyal Semantics

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Abstract

Łoś's theorem, also known as the fundamental result of ultraproducts, states that the ultraproduct over a family of structures for the same language satisfies a first-order formula if and only if the set of indices for which the structures satisfy the formula belongs to the underlying ultrafilter. The associated notion of satisfaction is the Tarskian one via the elements of the set-theoretic structure that allow interpreting the formula. In the context of topoi, Kripke-Joyal semantics extends Tarski's notion to categorical logic. In this article, we propose to extend Łoś's theorem to first-order structures on elementary topoi for Kripke-Joyal semantics. We also show that the extension entails its set-theoretic version. As is customary, we use the categorical version of Łoś's theorem to obtain a proof of the compactness theorem for Kripke-Joyal semantics.

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1 Introduction

Łoś's theorem is a result in model theory, also known as the fundamental result of ultraproducts. In its standard form, Łoś's theorem states that the ultraproduct on a family of structures for the same language satisfies a first-order formula if and only if the set of indices for which the structures satisfy the formula belongs to the underlying ultrafilter. In this article, we extend Łoś's theorem to categorical logic, more precisely to first-order structures on elementary topoi for Kripke-Joyal semantics. We also show that the extension entails its set-theoretic version. Finally, we use the categorical version of Łoś's theorem to obtain a categorical version of the compactness theorem.

In its set-based instance, Łoś's Theorem can be formulated as follows:

Theorem 1.1 (Łoś's Theorem [11]). *Let $(\mathcal{M}_i)_{i \in I}$ be an I -indexed family of nonempty Σ -structures, and let F be an ultrafilter on I . Let $\prod_F \mathcal{M}$ be the ultraproduct of $(\mathcal{M}_i)_{i \in I}$ with respect to F . Since each \mathcal{M}_i is nonempty, the ultraproduct $\prod_F \mathcal{M}$ is the quotient of $\prod_{i \in I} \mathcal{M}_i$ by the equivalence relation \sim_F identifying I -sequences that coincide on a set of indices¹ belonging to F . Let $(a_i^k)_{i \in I}$ be I -sequences for $k \in \{1, \dots, n\}$, with $[a^k]_{\sim_F}$ denoting their equivalence classes. Then for each Σ -formula φ ,*

$$\prod_F \mathcal{M} \models \varphi([a^1]_{\sim_F}, \dots, [a^n]_{\sim_F}) \text{ iff } \{j \in I \mid \mathcal{M}_j \models \varphi(a_j^1, \dots, a_j^n)\} \in F.$$

For instance, consider a family of structures $(\mathcal{M}_i)_{i \in \mathbb{N}}$ where each \mathcal{M}_i is a model of Peano's arithmetic. If F is a non-principal ultrafilter on \mathbb{N} , the ultraproduct $\prod_F \mathcal{M}$ is a model of Peano's arithmetic that provides a non-standard model of arithmetic containing "infinite" natural numbers. More applications to model theory, algebra, and non-standard analysis are discussed in the surveys by J. Keisler's [10] and S. Galbor [17]. In logic, and more precisely in model theory, the standard corollary of Łoś's theorem is the compactness theorem, stating that a set of first-order sentences admits a model if and only if every subset of it does.

¹That is $(a_i)_{i \in I} \sim_F (b_i)_{i \in I}$ if and only if $\{i \in I \mid a_i = b_i\} \in F$

The theorem considers a first-order signature $\Sigma = (S, F, R)$, i.e., consisting of a set of sorts, function symbols, and relation symbols, to be used to build Σ -structures. In set theory, these Σ -structures consist of sets associated with functions and relations characterized by the signature. Categorical logic [8, 14] uses tools from category theory to extend the set-theoretic semantics of first-order logic (FOL) to categories and, more specifically, to a family of categories known as elementary topoi or to specific fragments thereof (Cartesian, regular, coherent, Grothendieck's). A first difficulty occurs: in this framework, we can no longer talk about the elements of Σ -structure carriers as they are not sets anymore. In particular, I -sequences do not exist. Then, how can the ultraproduct $\prod_F \mathcal{M}$ be defined categorically? In the category of sets and functions, ultraproducts correspond to filtered products where the underlying filter is an ultrafilter. Thus, filtered products correspond to particular instances of the categorical concept of reduced products [2, 6, 7, 16], i.e., to colimits of directed diagrams of projections between the (direct) products determined by the corresponding filter.

The statement made in the theorem relies on the notion of satisfaction of a formula by a Σ -structure, which unveils a second difficulty: we need a categorical counterpart to this notion of satisfaction. In the set-theoretic framework, the usual Tarskian notion of satisfaction for a formula φ over Σ in a Σ -structure \mathcal{M} is given by the subset of the elements of the structure in which one can interpret the formula. More precisely, if φ has free variables among the sequence of typed variables $\bar{x} = (x_1 : s_1, \dots, x_n : s_n)$, the interpretation of φ is a subset S of $M_{\bar{x}} = M_{s_1} \times \dots \times M_{s_n}$ of values $\bar{a} = (a_i)_{i \in \{1, \dots, n\}}$, with $a_i \in M_{s_i}$, satisfying φ . The satisfaction of φ at the element \bar{a} is written $\mathcal{M} \models_{\bar{a}} \varphi$. Rephrasing the construction categorically in the category Set of sets and functions, the subset S is obtained via the following pullback:

$$\begin{array}{ccc} S & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \text{true} \\ M_{\bar{x}} & \xrightarrow{\chi_S} & \{0, 1\} \end{array}$$

where $\mathbf{1} = \{*\}$ is the terminal object in the category Set and χ_S is the characteristic function associated with the inclusion $S \subseteq M_{\bar{x}}$. The value \bar{a} then corresponds to a morphism $\mathbf{1} \rightarrow M_{\bar{x}}$, meaning that the satisfaction of φ at \bar{a} , i.e., $\mathcal{M} \models_{\bar{a}} \varphi$, is given by the commutativity of the following diagram:

$$\begin{array}{ccc} & & S \\ & \nearrow & \downarrow \\ \mathbf{1} & \xrightarrow{\bar{a}} & M_{\bar{x}} \end{array}$$

Kripke-Joyal semantics generalizes this pointwise interpretation to the categorical framework by replacing the morphisms $\bar{a} : \mathbf{1} \rightarrow M_{\bar{x}}$ with the **generalized elements** of $M_{\bar{x}}$, that is all morphisms $U \rightarrow M_{\bar{x}}$ (see [14, Sect. VI.6] or [9, Chap. 5, Sect. 4]). This interpretation leverages the fact that an object is determined (up to isomorphism) by its collection of generalized objects, as stated

by Yoneda’s lemma. Kripke-Joyal semantics also admits rules, sometimes called semantic rules, explaining how connectives and quantifiers are to be interpreted using the notion of generalized elements. These rules correspond to Def. 3.6. In this paper, they will be the key ingredient to proving Łoś’s Theorem for Kripke-Joyal semantics.

Finally, the set-theoretic definition of filtered products exploits that filters are closed under finite intersections (see the definition of a filter in Section 2). A third, more technical, difficulty then arises: we need some properties on the domains U of the generalized elements $U \rightarrow M_{\vec{x}}$ to perform the intersection as we no longer define filtered products pointwise. A solution is to require that the domains U are finitely presentable objects and satisfy the ascending chain conditions, which entails that they have finitely many subobjects.

The ultraproducts method has already been studied abstractly in (restrictions of) FOL [2, 6, 7]. To our knowledge, this method has not been explored within the framework of categorical logic that interprets formulae according to Kripke-Joyal semantics. Additionally, Makkai showed in [12] that any small pretopos \mathcal{C} can be reconstructed from its category of models, $Mod(\mathcal{C})$, using additional structure provided by the ultraproduct construction (ultracategories). From this construction, the classical set-theoretic version of Łoś’s theorem can be recovered by replacing the pretopos with the syntactic category of the appropriate first-order theory.

The paper is organized as follows. We introduce the notations used in the paper in Sect. 2 and recall some theoretical backgrounds about topoi and filters. In Sect. 3, we present Kripke-Joyal semantics, i.e., a categorical semantics for FOL, while Sect. 4 is dedicated to the main result of the paper, namely Łoś’s theorem, in the context of Kripke-Joyal semantics and its application to the compactness theorem.

2 Preliminaries

We assume familiarity with the main notions from category theory, such as categories, functors, natural transformations, limits, colimits, and Cartesian closedness. We refer the interested reader to standard textbooks such as [4, 13].

2.1 Notations

Throughout the paper, we write \mathcal{C} and \mathcal{D} for generic categories, X and Y for objects of categories, $\text{Ob}(\mathcal{C})$ for the collection of objects of a category \mathcal{C} , f, g , and h for morphisms, $\text{Hom}_{\mathcal{C}}(X, Y)$ for the hom-set from X to Y in \mathcal{C} , $F, G, H: \mathcal{C} \rightarrow \mathcal{D}$ for functors from a category \mathcal{C} into a category \mathcal{D} , and $\alpha, \beta: F \Rightarrow G$ for natural transformations. Given a morphism $f: X \rightarrow Y$, we write $\text{dom}(f) = X$ for the domain of f , $\text{cod}(f) = Y$ for its codomain, and $f: X \rightarrowtail Y$ if f is a monomorphism. For an object $X \in \text{Ob}(\mathcal{C})$, we write id_X for the identity morphisms on X . We write \emptyset and $\mathbf{1}$ for the initial and terminal objects, $X \times Y$ for the product of X and Y and $X + Y$ for their coproduct. Given a functor

$F: \mathcal{C} \rightarrow \mathcal{D}$, $F^{op}: \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$ is the opposite functor of F . Given two functors $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$, $F \dashv G$ means that F is left adjoint to G . Finally, when a category \mathcal{C} is Cartesian closed, X^Y denotes the exponential object of X and Y .

2.2 Subobjects and Heyting Algebras

Elementary topoi generalize the category of sets and functions, allowing a more abstract solution for logical reasoning. While many approaches can be taken to present and define topoi [8], we view them as a structure of intuitionistic logic where the notion of truth value is deeply linked with that of subobjects. We first recall the notion subobject before presenting topoi.

In a category \mathcal{C} , the set of subobjects $\text{Sub}(X)$ of an object X consists of the equivalence classes on the collections on mono into X , such that $f: A \rightarrow X$ and $g: B \rightarrow X$ are equivalent if and only if A and B are isomorphic. We write $[f]$ for the equivalence class of f . For instance, the subobjects in Set of a set X are the subsets of X (up to isomorphism). Additionally, $\text{Sub}(X)$ admits a partial order \leq_X such that $[f] \leq_X [g]$ if f factors through g , i.e., there is $h: A \rightarrow B$ such that $f = g \circ h$.

If \mathcal{C} is a finitely complete category such that the poset $\text{Sub}(X)$ is a small category, the mapping $S: X \mapsto \text{Sub}(X)$ yields a contravariant functor $\text{Sub}: \mathcal{C}^{op} \rightarrow \text{Pos}$, with Pos being the category of posets. In addition to mapping objects X of \mathcal{C} to $\text{Sub}(X)$, Sub maps morphisms $f: X \rightarrow Y$ to base change functors $f^*: \text{Sub}(Y) \rightarrow \text{Sub}(X)$. Given $f: X \rightarrow Y$, the base change functor f^* maps each $[Y' \rightarrow Y]$ to $[X' \rightarrow X]$, making the following diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

a pullback.

For logical purposes, we want more structure than just posets for $\text{Sub}(X)$. Typically, we want $\text{Sub}(X)$ to be at least a Heyting algebra, i.e., a distributive bounded lattice that admits an implication \rightarrow such that for any A in $\text{Sub}(X)$, the following adjunction holds $(- \wedge A) \dashv (A \rightarrow -)$. The largest and smallest objects of $(\text{Sub}(X), \leq_X)$ are respectively $[\text{id}_X]$ and $[\emptyset \rightarrow X]$.

When there is no ambiguity on the codomain X , we may write Y instead of $[Y \rightarrow X]$ for a subobject.

2.3 Elementary Topoi

Elementary topoi are categories that allow equipping the poset $\text{Sub}(X)$ with such a structure of Heyting algebra. Topoi are finitely complete Cartesian closed category with a subobject classifier [8, Chap. A2]. A subobject classifier is a monomorphism $\text{true}: \mathbf{1} \rightarrow \Omega$ out of the terminal object $\mathbf{1}$, such that for every

monomorphism $m : Y \rightarrow X$, there exists a unique morphism $\chi_m : X \rightarrow \Omega$ such that the following diagram is a pullback:

$$\begin{array}{ccc} Y & \xrightarrow{!} & \mathbf{1} \\ m \downarrow & & \downarrow \text{true} \\ X & \xrightarrow{\chi_m} & \Omega. \end{array}$$

The morphism χ_m is called the **characteristic** or **classifying morphism** of m . Hence, Ω represents the contravariant functor Sub , i.e., for every $X \in \text{Ob}(\mathcal{C})$, $\text{Sub}(X) \simeq \text{Hom}_{\mathcal{C}}(X, \Omega)$, and the universal object is $\text{true} : \mathbf{1} \rightarrow \Omega$.

A topos also fulfills the following properties [3, 8]:

- It is finitely cocomplete.
- It has an initial object \emptyset and a terminal object $\mathbf{1}$, which are the colimit and limit of the empty diagram (since it is finitely complete and cocomplete).
- Epimorphisms and monomorphisms form a factorization system, i.e., every morphism f can be uniquely factorized as $m_f \circ e_f$ where e_f is an epimorphism and m_f is a monomorphism. The codomain of e_f is called the **image of f** and written $\text{Im}(f)$. Then $(A \xrightarrow{f} B) = (A \xrightarrow{e_f} \text{Im}(f) \xrightarrow{m_f} B)$.
- Every object X in $\text{Ob}(\mathcal{C})$ has a **power object** PX defined as the exponential Ω^X . Power objects generalize the notion of powerset from the category Set . As a topos is Cartesian closed, a power object satisfies the following adjunction:

$$\text{Hom}_{\mathcal{C}}(X \times Y, \Omega) \simeq \text{Hom}_{\mathcal{C}}(X, PY).$$

Topoi encompass Set , i.e., the category of sets, Grothendieck topoi, i.e., categories equivalent to the category of sheaves over a site [5], categories of presheaves, i.e., the categories $\widehat{\mathcal{C}}$ of functors $F : \mathcal{C}^{op} \rightarrow \text{Set}$, where \mathcal{C} is required to be small. Interestingly, presheaf topoi subsume most algebraic structures used in computer science, like graphs, hypergraphs, and simplicial sets.

In a topos, the poset of subobjects $\text{Sub}(X)$ is a Heyting algebra [8]. Additionally, each base change functor f^* admits both left and right adjoints \exists_f and \forall_f , i.e., $\exists_f \dashv f^* \dashv \forall_f$.

2.4 Filters, Filtered Products, and Filtered Colimits

2.4.1 Set-theoretic Filters and Ultrafilters

Given a nonempty set I , a **filter F over I** is a subset of $\wp(I)$ such that:²

- $I \in F$;

² $\wp(I)$ denotes the powerset of I .

- if $A, B \in F$, then $A \cap B \in F$, and
- if $A \in F$ and $A \subseteq B$, then $B \in F$.

For instance, $\{I\}$ and $\wp(I)$ are filters on I . The filter generated by some $J \subseteq I$ is $F_J = \{A \in \wp(I) \mid J \subseteq A\}$. It is called a **principal filter**. If I is finite, all filters on I are principal.

A filter is an **ultrafilter** if it is maximal for inclusion. In particular, if U is an ultrafilter, then every $A \in \wp(I)$ is in U if and only if $I \setminus A$ is not in U . By Zorn's lemma, any filter is contained in an ultrafilter.

2.4.2 Filtered Products in Categories

In Set, filtered products correspond to directed colimits of products of sets, which have then been extended to arbitrary categories [16], leading to colimits of directed diagrams of projections between the (direct) products determined by the corresponding filter [2, 6, 7].

Definition 2.1 (Filtered product). *Let F be a filter over a set of indices I , and let $X = (X_i)_{i \in I}$ be a I -indexed family of objects in \mathcal{C} . Then, F and X induce a functor $A_F: F^{op} \rightarrow \mathcal{C}$, mapping each subset inclusion $J \subseteq J'$ of F to the canonical projection $p_{J', J}: \prod_{J'} X_j \rightarrow \prod_J X_j$.*

*The **filtered product** of X modulo F is the colimit $\mu: A_F \Rightarrow \prod_F X$ of the functor A_F . \mathcal{C} **have filtered products** if any filter F and any I -indexed family of objects $X = (X_i)_{i \in I}$ in \mathcal{C} yield a filtered product of X modulo F .*

Filtered products are unique up to isomorphisms since they are colimits, and we can talk about *the* filtered product of X modulo F .

For instance, Set have filtered products, and, therefore, any presheaf topos $\widehat{\mathcal{B}}$ for some small category \mathcal{B} also have them. Indeed, presheaf limits and colimits are computed componentwise, meaning that filtered products can be lifted from sets to presheaves.

2.4.3 Locally Finitely Presentable Category

Filtered products correspond to colimits where the underlying diagram is a filtered category.

Definition 2.2 (Filtered category). *A **filtered category** is a category \mathcal{C} in which every finite diagram has a cocone.*

Definition 2.3 (Filtered colimit). *A **filtered colimit** is a colimit of a functor $D: \mathcal{J} \rightarrow \mathcal{C}$ where the shape \mathcal{J} of D is a filtered category.*

Note that filtered products are filtered colimits rather than filtered limits (as the name would suggest). We are interested in filtered colimits to define finitely presentable objects, also called finitely presented [6] or compact.

Definition 2.4 (Finitely presentable object). *An object X of a category \mathcal{C} is **finitely presentable** if the hom-functor $\text{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \rightarrow \text{Set}$ preserves filtered colimits.*

The definition of finitely presentable objects means that, for any functor $D : \mathcal{J} \rightarrow \mathcal{C}$ where \mathcal{J} is a filtered category, a morphism $\mu : X \rightarrow \text{colim}(D)$ factors essentially uniquely through some $\nu_i : D(i) \rightarrow \text{colim}(D)$. More precisely, the definition is equivalent to the following condition:

- for every morphism $\mu : X \rightarrow V$ to the vertex of a colimiting co-cone $\nu : D \rightarrow V$ of a directed diagram $D : (I, \leq) \rightarrow \mathcal{C}$, there exists $i \in I$ and a morphism $\mu_i : X \rightarrow D(i)$ such that $\mu = \nu_i \circ \mu_i$, and
- for any two morphisms μ_i and μ_j as above, there exists k such that $k > i$, $k > j$, and $D_{i,k} \circ \mu_i = D_{j,k} \circ \mu_j$.

Definition 2.5 (Locally finitely presentable category [1]). *A locally small category³ \mathcal{C} is **locally finitely presentable** if*

- *it has all small limits (i.e., is complete),*
- *it has a set A of finitely presentable objects, called **generators**, such that every object in \mathcal{C} is a filtered colimit of objects in A .*

Examples may be consulted in [1]. For our purposes, locally finitely presentable topoi encompass presheaves, atomic, and coherent topoi [8, Chap. D3, Sect. 3].

Note that locally finitely presentable categories also have all small colimits because any object is obtained as the filtered colimit of generators.

2.5 Noetherian Objects

While finitely presentable objects and locally finitely presentable categories ensure a controlled, finite generation process, Noetherian objects look at finiteness via subobject chains, ensuring that ascending sequences stabilize.

Definition 2.6 (Noetherian object). *An object X of a category \mathcal{C} is **Noetherian** if the set of its subobjects satisfies the ascending chain condition, i.e., if every sequence $X_0 \leq_X X_1 \leq_X \dots$ of subobjects of X eventually becomes stationary.*

Noetherian objects are finitary (have finitely many subobjects) in cocomplete categories. More precisely, the following result holds:

Proposition 2.7. *Let X be an object in a category \mathcal{C} such that X is Noetherian and $\text{Sub}(X)$ has arbitrary coproducts. Then $\text{Sub}(X)$ is finite.*

³A category whose hom-sets are sets.

Proof. Let $S_0 = \emptyset \subseteq S_1 \subseteq S_2 \subseteq \dots$ be an ascending chain in $\wp(\text{Sub}(X))$ where each S_{i+1} is obtained from S_i by adding to it a finite number of subobjects (for $i \geq 0$). Since $\text{Sub}(X)$ has arbitrary coproducts, we can consider, for each $i \geq 0$, the subobject $\bigvee S_i$ corresponding to the coproduct of the subobjects in S_i . By construction, we have an ascending chain $\bigvee S_0 \leq_X \bigvee S_1 \leq_X \bigvee S_2 \leq_X \dots$. Because X is Noetherian, this ascending chain is stationary. Since the upper limit of this chain is X , we conclude that $\text{Sub}(X)$ is necessarily finite. \square

Restricting locally finitely presentable topoi such that finitely presentable objects are Noetherian yields atomic topoi, some coherent topoi, such as sheaves on a Noetherian topological space, and presheaf topoi on a small Noetherian category, such as sets, graphs, and hypergraphs.

3 Categorical Semantics for First-Order Logic (FOL)

Following the seminal work of Alfred Tarski [19, 18], it is well-established that FOL can be interpreted with set-theoretic models where sorts are interpreted as sets, function symbols as functions between sets, and relations symbols as subsets of Cartesian products. Categorical logic builds on this construction, interpreting sorts as objects, terms as morphisms, and formulae as subobjects. In this section, we consider an elementary topos \mathcal{C} .

3.1 The Category of Σ -Structures

In the sequel, we consider a FOL-signature $\Sigma = (S, F, R)$, where S is the set of sorts, F the set of function symbols, and R the set of relation symbols. We first recall the notion of Σ -structures in a topos [8, Chap. D1].

Definition 3.1 (Σ -structure). *A Σ -structure \mathcal{M} in \mathcal{C} is defined by:*

- an object $M_s \in \text{Ob}(\mathcal{C})$ for every sort s in S ,
- a morphism $f^{\mathcal{M}} : M_{s_1} \times \dots \times M_{s_n} \rightarrow M_s \in \mathcal{C}$ for every function symbol f in F with profile $s_1 \times \dots \times s_n \rightarrow s$, and $f^{\mathcal{M}} : \mathbf{1} \rightarrow M_s$ if f is a constant symbol and $n = 0$,
- a subobject $r^{\mathcal{M}} \in \text{Sub}(M_{s_1} \times \dots \times M_{s_n})$ for every relation symbol r in R with profile $s_1 \times \dots \times s_n$.

A Σ -structure morphism $h : \mathcal{M} \rightarrow \mathcal{N}$ in \mathcal{C} is a family of morphisms $(h_s : M_s \rightarrow N_s)_{s \in S}$ such that:

- the diagram

$$\begin{array}{ccc} M_{s_1} \times \dots \times M_{s_n} & \xrightarrow{f^{\mathcal{M}}} & M_s \\ h_{s_1} \times \dots \times h_{s_n} \downarrow & & \downarrow h_s \\ N_{s_1} \times \dots \times N_{s_n} & \xrightarrow{f^{\mathcal{N}}} & N_s \end{array}$$

commutes, for every function symbol $f : s_1 \times \dots \times s_n \rightarrow s$ in F .

- there is a morphism $O \rightarrow O'$ such that the diagram

$$\begin{array}{ccc} O & \xrightarrow{r^{\mathcal{M}}} & M_{s_1} \times \dots \times M_{s_n} \\ \downarrow & & \downarrow \mu_{s_1} \times \dots \times \mu_{s_n} \\ O' & \xrightarrow{r^{\mathcal{N}}} & N_{s_1} \times \dots \times N_{s_n} \end{array}$$

commutes, for every relation symbol $r : s_1 \times \dots \times s_n$ in R .

Σ -structures and Σ -structure morphisms in \mathcal{C} form a category written $\Sigma\text{-Str}(\mathcal{C})$.

Proposition 3.2. *The category $\Sigma\text{-Str}(\mathcal{C})$ has small products.*

Proof. Let I be a set and $(\mathcal{M}_i)_{i \in I}$ be an I -indexed family of models. Consider the model $\prod_I \mathcal{M}_i$ defined by:

- for every $s \in S$, $(\prod_I \mathcal{M}_i)_s = \prod_I (M_i)_s$, with $\prod_I (M_i)_s$ being the product in \mathcal{C} ,
- for every $f : s_1 \times \dots \times s_n \rightarrow s \in F$, $f^{\prod_I \mathcal{M}_i}$ is the unique morphism such that the following diagram

$$\begin{array}{ccc} (\prod_I \mathcal{M}_i)_{s_1} \times \dots \times (\prod_I \mathcal{M}_i)_{s_n} & \xrightarrow{f^{\prod_I \mathcal{M}_i}} & (\prod_I \mathcal{M}_i)_s \\ \begin{array}{c} \downarrow \\ (p_{I,i})_{s_1} \times \dots \times (p_{I,i})_{s_n} \end{array} & & \downarrow (p_{I,i})_s \\ (M_i)_{s_1} \times \dots \times (M_i)_{s_n} & \xrightarrow{f_i^{\mathcal{M}}} & (M_i)_s \end{array}$$

commutes for all $i \in I$, which is well-defined by the universal property of small products in \mathcal{C} ,

- for every $r : s_1 \times \dots \times s_n \in R$, $r^{\prod_I \mathcal{M}_i}$ is the subobject $\prod_I O_i \twoheadrightarrow (\prod_I \mathcal{M}_i)_{s_1} \times \dots \times (\prod_I \mathcal{M}_i)_{s_n}$ where $r^{\mathcal{M}_i} : O_i \twoheadrightarrow (M_i)_{s_1} \times \dots \times (M_i)_{s_n}$.

Since each $(\prod_I \mathcal{M}_i)_s$ for $s \in S$ is obtained as a small product in \mathcal{C} , it follows that $\prod_I \mathcal{M}_i$ is the small product of $(\mathcal{M}_i)_{i \in I}$. \square

If x is a Σ -variable of sort s , we write $x : s$. We also write \vec{x} for a sequence of variables and $\vec{x}_1 \vec{x}_2$, resp. $\vec{x}_1 y$, when concatenating such sequences, resp. appending a new variable. These two notations naturally extend to terms. We say that a sequence of variables $\vec{x} = (x_1 : s_1, \dots, x_n : s_n)$ is a **suitable context** for a Σ -term, resp. a Σ -formula, if all free variables of this term, resp. formula, belong to $\{x_1, \dots, x_n\}$. We write $\vec{x}.t$, resp. $\vec{x}.\varphi$, to denote that \vec{x} is a suitable context for t , resp. φ . Then $\vec{x}.t$, resp. $\vec{x}.\varphi$, is called a **term-in-context**, resp. a **formula-in-context**. Additionally, we write $M_{\vec{x}}$ instead of $M_{s_1} \times \dots \times M_{s_n}$ for a sequence of variables $\vec{x} = (x_1 : s_1, \dots, x_n : s_n)$.

Given a Σ -structure \mathcal{M} , the **interpretation** of a term-in-context $\vec{x}.t$ in \mathcal{M} , with $t : s$, is a morphism $[[\vec{x}.t]]_{\mathcal{M}} : M_{\vec{x}} \rightarrow M_s$, defined inductively as follows:

- if t is a variable, then it corresponds to an x_j in \vec{x} , and $[[\vec{x}.x_j]]_{\mathcal{M}}$ is the projection on the sort associated with x_j ;
- if t is $f(t_1, \dots, t_n)$ for some function symbol f and $t_1 : s'_1, \dots, t_n : s'_n$, then $[[\vec{x}.t]]_{\mathcal{M}}$ is the composite

$$M_{\vec{x}} \xrightarrow{([[\vec{x}.t_1]]_{\mathcal{M}}, \dots, [[\vec{x}.t_n]]_{\mathcal{M}})} M_{s'_1} \times \dots \times M_{s'_n} \xrightarrow{f^{\mathcal{M}}} M_s.$$

The interpretation of Σ -terms in \mathcal{M} extends to the interpretation of formulae-in-context $\vec{x}.\varphi$ in \mathcal{M} , written $[[\vec{x}.\varphi]]_{\mathcal{M}}$ and defined inductively as follows:

- if φ is an equation $t = t'$, then $[[\vec{x}.(t = t')]]_{\mathcal{M}}$ is the equalizer of

$$M_{\vec{x}} \begin{array}{c} \xrightarrow{[[\vec{x}.t]]_{\mathcal{M}}} \\ \xrightarrow{[[\vec{x}.t']]_{\mathcal{M}}} \end{array} M_s$$

where s is the common sort of t and t' ;

- if φ is a relation $r(\vec{t})$ with $r : s_1 \times \dots \times s_n$, then $[[\vec{x}.r(\vec{t})]]_{\mathcal{M}}$ is the subobject $O' \rightrightarrows M_{\vec{x}}$ obtained from the following pullback:

$$\begin{array}{ccc} O' & \longrightarrow & O \\ [[\vec{x}.r(\vec{t})]]_{\mathcal{M}} \downarrow & & \downarrow r^{\mathcal{M}} \\ M_{\vec{x}} & \xrightarrow{[[\vec{x}.\vec{t}]]_{\mathcal{M}}} & M_{s_1} \times \dots \times M_{s_n} \end{array}$$

- if φ is a formula of the form $\psi \wedge \chi$, $\psi \vee \chi$, or $\psi \Rightarrow \chi$, then $[[\vec{x}.\varphi]]_{\mathcal{M}}$ is interpreted via the operators \wedge , \vee , and \rightarrow of the Heyting algebra $\text{Sub}(M_{\vec{x}})$.
- if φ is a formula of the form $\exists y.\psi$, with $y : s$, then $[[\vec{x}.(\exists y.\psi)]]_{\mathcal{M}} = \exists_{\pi} [[\vec{x}y.\psi]]_{\mathcal{M}}$ with π being the projection $: M_{\vec{x}} \times M_s \rightarrow M_{\vec{x}}$ and $\exists_{\pi} \dashv \pi^*$.
- if φ is a formula of the form $\forall y.\psi$ then $[[\vec{x}.(\forall y.\psi)]]_{\mathcal{M}} = \forall_{\pi} [[\vec{x}y.\psi]]_{\mathcal{M}}$, with π being the same projection and $\forall_{\pi} \dashv \pi^*$.

Note that, by construction, the interpretation $[[\vec{x}.\varphi]]_{\mathcal{M}}$ of the formula-in-context $\vec{x}.\varphi$ in \mathcal{M} is a subobject of $M_{\vec{x}}$. We write $\{\vec{x} \mid \varphi(\vec{x})\}_{\mathcal{M}}$ for its domain, i.e., $[[\vec{x}.\varphi]]_{\mathcal{M}} : \{\vec{x} \mid \varphi(\vec{x})\}_{\mathcal{M}} \rightrightarrows M_{\vec{x}}$.

3.2 Kripke-Joyal Semantics

In most categories, the elements of an object do not correspond to a well-defined notion. Therefore, the notion of elements of a Σ -structure needs to be replaced to redefine the satisfaction of FOL formulae. A solution is to consider the generalized elements of \mathcal{C} , thus regarding \mathcal{C} as its image under the Yoneda embedding. Indeed, by Yoneda's lemma, an object X is uniquely determined, up to isomorphisms, by the functor $\text{Hom}_{\mathcal{C}}(-, X)$, i.e., the morphisms $Y \rightarrow X$ in

\mathcal{C} , also called **generalized elements** [5]. In topoi, the fundamental theorem, or slice theorem, [15] ensures that the generalized elements of X in the topos \mathcal{C} correspond to ordinary points in the slice topos \mathcal{C}/X .

In this subsection, we consider an object \mathcal{M} of $\Sigma\text{-Str}(\mathcal{C})$, i.e., a Σ -structure in \mathcal{C} , a Σ -formula φ with a suitable context $\vec{x} = (x_1 : s_1, \dots, x_n : s_n)$, and a generalized element $\alpha : U \rightarrow M_{\vec{x}}$.

Kripke-Joyal semantics is thoroughly explained in [14, Sect. VI.6], with different notations and using the forcing relation $U \Vdash \varphi(\alpha)$ between the generalized element and the formula, instead of a relation between the Σ -structure and the formula. More compact explanations might be found in [9, Chap. 5, Sect. 4], again using different notations.

Definition 3.3 (Kripke-Joyal semantics). *The satisfaction of $\vec{x}.\varphi$ in \mathcal{M} by the generalized element α , written $\mathcal{M} \models_{\alpha} \vec{x}.\varphi$, is defined as:*

$$\mathcal{M} \models_{\alpha} \vec{x}.\varphi \text{ iff } \alpha \text{ factors through } \llbracket \vec{x}.\varphi \rrbracket_{\mathcal{M}}$$

i.e., the following diagram commutes:

$$\begin{array}{ccc} & \{\vec{x} \mid \varphi(\vec{x})\}_{\mathcal{M}} & \longrightarrow \mathbf{1} \\ & \downarrow \llbracket \vec{x}.\varphi \rrbracket_{\mathcal{M}} & \downarrow \text{true} \\ U & \xrightarrow{\alpha} M_{\vec{x}} & \xrightarrow{\varphi(x)} \Omega. \end{array}$$

Equivalently, this means that $\text{Im}(\alpha) \leq \{\vec{x} \mid \varphi(\vec{x})\}_{\mathcal{M}}$.

As discussed in [14, Chap. VI, Sect. 6], Kripke-Joyal semantics satisfies the following two properties, called **monotonicity** and **local character**.

Proposition 3.4 (Monotonicity [14, Chap.VI, Sect.6]). *If $\mathcal{M} \models_{\alpha} \vec{x}.\varphi$, then, for any morphism $f : V \rightarrow U$ in \mathcal{C} , $\mathcal{M} \models_{\alpha \circ f} \vec{x}.\varphi$.*

Proposition 3.5 (Local character [14, Chap.VI, Sect.6]). *If $f : V \rightarrow U$ is an epimorphism and $\mathcal{M} \models_{\alpha \circ f} \vec{x}.\varphi$, then $\mathcal{M} \models_{\alpha} \vec{x}.\varphi$.*

These two properties are the key to proving the following theorem, allowing for an inductive description of Kripke-Joyal semantics. This inductive description will later turn useful as the proof of Łoś's theorem is conducted by induction. This theorem is also known as the semantic rules of Kripke-Joyal semantics.

Theorem 3.6 (Theorem VI.6.1 in [14]). *Let $\alpha : U \rightarrow M_{\vec{x}}$ be a generalized element, φ and ψ be Σ -formulae, then*

- $\mathcal{M} \models_{\alpha} \vec{x}.\varphi \wedge \psi$ iff $\mathcal{M} \models_{\alpha} \vec{x}.\varphi$ and $\mathcal{M} \models_{\alpha} \vec{x}.\psi$;
- $\mathcal{M} \models_{\alpha} \vec{x}.\varphi \vee \psi$ iff there are morphisms $p : V \rightarrow U$ and $q : W \rightarrow U$ such that $p + q : V + W \rightarrow U$ is an epimorphism, and both $\mathcal{M} \models_{\alpha \circ p} \vec{x}.\varphi$ and $\mathcal{M} \models_{\alpha \circ q} \vec{x}.\psi$;
- $\mathcal{M} \models_{\alpha} \vec{x}.\varphi \Rightarrow \psi$ iff for any morphism $p : V \rightarrow U$ such that $\mathcal{M} \models_{\alpha \circ p} \vec{x}.\varphi$, then $\mathcal{M} \models_{\alpha \circ p} \vec{x}.\psi$;

- $\mathcal{M} \models_{\alpha} \tilde{x}.\neg\varphi$ iff for any morphism $p: V \rightarrow U$ such that $\mathcal{M} \models_{\alpha \circ p} \tilde{x}.\varphi$, then $V \simeq \emptyset$;

For the quantifiers, we consider an additional variable $y: s$. Then

- $\mathcal{M} \models_{\alpha} \tilde{x}.\langle \exists y.\varphi \rangle$ iff there exists an epimorphism $p: V \rightarrow U$ and a generalized element $\beta: V \rightarrow M_s$ such that $\mathcal{M} \models_{(\alpha \circ p, \beta)} \tilde{x}y.\varphi$;
- $\mathcal{M} \models_{\alpha} \tilde{x}.\langle \forall y.\varphi \rangle$ iff for every morphism $p: V \rightarrow U$ and every generalized element $\beta: V \rightarrow M_s$, it holds that $\mathcal{M} \models_{(\alpha \circ p, \beta)} \tilde{x}y.\varphi$;

Kripke-Joyal semantics provide a notion of satisfaction relative to a generalized element, which can be aggregated into a global notion.

Definition 3.7 (Model). \mathcal{M} is a **model** for $\tilde{x}.\varphi$, written $\mathcal{M} \models \tilde{x}.\varphi$, if for all generalized elements $\alpha: U \rightarrow M_{\tilde{x}}$, $\mathcal{M} \models_{\alpha} \tilde{x}.\varphi$.

Proposition 3.8. \mathcal{M} is a model for $\tilde{x}.\varphi$ if and only if $\mathcal{M} \models_{Id_{M_{\tilde{x}}}} \tilde{x}.\varphi$.

Proof. The implication is obvious, and the converse follows from monotonicity (Proposition 3.4). \square

When the topos is locally finitely presentable, we can restrict the study to finitely presentable generalized objects.

Proposition 3.9. If \mathcal{C} is a locally finitely presentable elementary topos, then $\mathcal{M} \models \tilde{x}.\varphi$ if and only if for all generalized elements $\alpha: U \rightarrow M_{\tilde{x}}$ such that U is finitely presentable $\mathcal{M} \models_{\alpha} \tilde{x}.\varphi$.

Proof. The direct implication is obvious. For the converse, consider a generalized element $\alpha: U \rightarrow M_{\tilde{x}}$. Since \mathcal{C} is a locally finitely presentable category, U is a filtered colimit of finitely presentable objects $(A_i)_{i \in I}$ (see Definition 2.5). For $i \in I$, we write $\nu_i: A_i \rightarrow U$ for the coprojection of the filtered colimit. Then, $\alpha \circ \nu_i: A_i \rightarrow M_{\tilde{x}}$ is a generalized elements and A_i is finitely presentable. By hypothesis, $\mathcal{M} \models_{\alpha \circ \nu_i} \tilde{x}.\varphi$, i.e., $\alpha \circ \nu_i$ factors through $[[\tilde{x}.\varphi]]_{\mathcal{M}}$ (see Definition 3.3). By the universal property of colimit, there is a unique morphism $U \rightarrow \{\tilde{x} \mid \varphi(\tilde{x})\}_{\mathcal{M}}$ such that the obvious diagram commutes. In particular, α factors through $[[\tilde{x}.\varphi]]_{\mathcal{M}}$, i.e., $\mathcal{M} \models_{\alpha} \tilde{x}.\varphi$. From Definition 3.7, we conclude that $\mathcal{M} \models \tilde{x}.\varphi$. \square

Proposition 3.9 can be restricted to monomorphisms.

Corollary 3.10. If \mathcal{C} is a locally finitely presentable elementary topos, then $\mathcal{M} \models \tilde{x}.\varphi$ if and only if for all monomorphisms $\alpha: U \rightarrow M_{\tilde{x}}$ such that U is finitely presentable $\mathcal{M} \models_{\alpha} \tilde{x}.\varphi$.

Proof. This follows from epi-mono factorization, local character (see Proposition 3.5), and that X is a finitely presentable object and $f: X \rightarrow Y$ is an epimorphism, then Y is also finitely presentable. \square

3.3 Internal Logic of a Topos

Kripke-Joyal semantics is often used in conjunction with the Mitchell-Bénabou language, which considers, given a topos \mathcal{C} , the signature $\Sigma_{\mathcal{C}}$ having sorts $[X]$ for all objects X in \mathcal{C} , function names $[f]$ for all morphisms f in \mathcal{C} , and relations names $[r]$ for all monomorphisms r in \mathcal{C} . By mapping $[X]$ to X , $[f]$ to f and $[r]$ to r , we obtain a canonical $\Sigma_{\mathcal{C}}$ -structure called the **tautological** $\Sigma_{\mathcal{C}}$ -structure [5, Chap. 1, Sect. 5.2.1]. More precisely, for every topos \mathcal{C} , we can define an internal language $\mathcal{L}_{\mathcal{C}}$ composed of types defined by the objects of \mathcal{C} , from which we can define terms as follows:

- $true : X$;
- $x : X$ where x is a variable and X is a type;
- $f(t) : Y$ where $f : X \rightarrow Y$ is a morphism of \mathcal{C} and $t : X$ is a term;
- $\langle t_1, \dots, t_n \rangle : X_1 \times \dots \times X_n$ if for every i , $1 \leq i \leq n$, $t_i : X_i$ is a term;
- $(t)_i : X_i$ if $t : X_1 \times \dots \times X_n$ is a term;
- $\{x : X \mid \alpha\} : PX$ if $\alpha : \Omega$ is a term;
- $\sigma = \tau : \Omega$ if σ and τ are terms of the same type;
- $\sigma \in_X \tau : \Omega$ if $\sigma : X$ and $\tau : PX$ are terms;
- $\sigma \leq_X \tau : \Omega$ if $\sigma, \tau : PX$ are terms;
- $\varphi @ \psi : \Omega$ if $\varphi : \Omega$ and $\psi : \Omega$ are terms with $@ \in \{\wedge, \vee, \Rightarrow\}$;
- $\neg\varphi : \Omega$ if $\varphi : \Omega$ is a term;
- $Qx. \varphi : \Omega$ if $x : X$ and $\varphi : \Omega$ are terms and $Q \in \{\forall, \exists\}$.

Additionally, **formulae** correspond to terms of type Ω . This internal language allows for reasoning about \mathcal{C} as if it were a set, using simple term expressions. We will use this internal language to demonstrate some results in this paper.

4 Filtered Products in $\Sigma\text{-Str}(\mathcal{C})$ and Łoś's Theorem

4.1 Filtered Products in $\Sigma\text{-Str}(\mathcal{C})$

We now impose the elementary topos \mathcal{C} to have filtered products (such that we can consider ultraproducts). Additionally, we consider a set I , a filter F over it and a family of Σ -structures $(\mathcal{M}_i)_{i \in I}$. The filtered product of \mathcal{M} modulo F is the colimit $\mu : A_F \Rightarrow \prod_F M$ of the functor A_F as defined in Definition 2.1. Under some conditions discussed in Proposition 4.1, it corresponds to the Σ -structure $\prod_F \mathcal{M}$ of $(\mathcal{M}_i)_{i \in I}$ defined as follows:

- for every $s \in S$, $(\prod_F M)_s$ is the filtered product of $((M_i)_s)_{i \in I}$.
- for every function name $f: s_1 \times \dots \times s_n \rightarrow s$, $f^{\prod_F \mathcal{M}}$ is the unique morphism such that the following diagram commutes, according to the universal property of colimits:

$$\begin{array}{ccc}
(\prod_I M_i)_{s_1} \times \dots \times (\prod_I M_i)_{s_n} & \xrightarrow{f^{\prod_I \mathcal{M}_i}} & (\prod_I M_i)_s \\
(\mu_I)_{s_1} \times \dots \times (\mu_I)_{s_n} \downarrow & & \downarrow (\mu_I)_s \\
(\prod_F M)_{s_1} \times \dots \times (\prod_F M)_{s_n} & \xrightarrow{f^{(\prod_F \mathcal{M})}} & (\prod_F M)_s.
\end{array}$$

- for every relation name $r: s_1 \times \dots \times s_n$, $r^{\prod_F \mathcal{M}}$ is the filtered product of the family $(r^{\mathcal{M}_i})_{i \in I}$.

We introduce two propositions about filtered products that will be useful for proving Loś's theorem. The first proposition claims that the Σ -structure $\prod_F \mathcal{M}$ is indeed the filtered product if the projections associated with the functor A_F are epimorphisms.

Proposition 4.1. *Under the condition that the projections $p_{I,J}: \prod_I \mathcal{M}_i \rightarrow \prod_J \mathcal{M}_j$ are epimorphisms for all subsets J of I in F , $\prod_F \mathcal{M}$ is the filtered product of $(\mathcal{M}_i)_{i \in I}$ modulo F .*

First, we introduce a lemma stating that under the condition of Proposition 4.1, the μ_J are also epimorphisms.

Lemma 4.2. *Under the condition that the projections $p_{I,J}: \prod_I \mathcal{M}_i \rightarrow \prod_J \mathcal{M}_j$ are epimorphisms for all subsets $J \in F$, then so are all the $\mu_J: \prod_J \mathcal{M}_j \rightarrow \prod_F \mathcal{M}$ for all the subsets $J \in F$.*

Proof. For $J \in F$, the set $F|_J = \{J \cap K \mid K \in F\}$ is still a filter⁴. Moreover, $\prod_{F|_J} \mathcal{M}$ and $\prod_F \mathcal{M}$ are isomorphic (see Proposition 6.3 in [6]). Since $p_{J,K}$ is an epimorphism for every $K \in F|_J$ and $(\mu_K)_{K \in F|_J}$ is a jointly epic family (because $\prod_{F|_J} \mathcal{M}$ is a filtered product), μ_J is an epimorphism. \square

We can now prove Proposition 4.1.

Proof. The family $\mu = (\mu_J)_{J \in F}$ forms a cocone $A_F \Rightarrow \prod_F \mathcal{M}$ where the functor $A_F: F \rightarrow \Sigma\text{-Str}(\mathcal{C})$ maps indices J of the filter F to models $\prod_J \mathcal{M}_j$ and inclusions $J \subseteq J'$ to projections $p_{J',J}$.

Let $\nu: A_F \Rightarrow \mathcal{N}$ be another cocone. Since the category \mathcal{C} has filtered products, there is a unique morphism $\theta_s: (\prod_F M)_s \rightarrow N_s$, for every sort s in S .

We now verify that the family $\theta = (\theta_s)_{s \in S}$ of morphisms in \mathcal{C} form a valid morphism in $\Sigma\text{-Str}(\mathcal{C})$. In other words, we need to show that $\theta \circ f^{\prod_F \mathcal{M}} = f^{\mathcal{N}} \circ \theta$, resp. $\theta \circ r^{\prod_F \mathcal{M}} = r^{\mathcal{N}} \circ \theta$ for all function symbols, resp. relation symbols, in Σ .

⁴Note that $J \neq \emptyset$ as F is an ultrafilter is necessary to ensure that $F|_J$ is still a filter.

We recall that for every $J \in F$, and every $x : \prod_J (M_j)_{s_1} \times \dots \times \prod_J (M_j)_{s_n}$, we have that $\theta(\mu_J(x)) = \nu_J(x)$.

First, consider a function name $f : s_1 \times \dots \times s_n \rightarrow s$ in F . We aim to prove that $\theta \circ f^{\prod_F \mathcal{M}} = f^{\mathcal{N}} \circ \theta$, which amounts to showing that the following diagram commutes.

$$\begin{array}{ccc}
 (\prod_I M_i)_{s_1} \times \dots \times (\prod_I M_i)_{s_n} & \xrightarrow{f^{\prod_I \mathcal{M}_i}} & (\prod_I M_i)_s \\
 \downarrow ((\mu_I)_{s_1}, \dots, (\mu_I)_{s_n}) & & \downarrow (\mu_I)_s \\
 (\nu_I)_{s_1}, \dots, (\nu_I)_{s_n} \left((\prod_F M)_{s_1} \times \dots \times (\prod_F M)_{s_n} \right. & \xrightarrow{f^{\prod_F \mathcal{M}}} & \left. (\prod_F M)_s \right) (\nu_I)_s \\
 \downarrow ((\theta_I)_{s_1}, \dots, (\theta_I)_{s_n}) & & \downarrow (\theta_I)_s \\
 N_{s_1} \times \dots \times N_{s_n} & \xrightarrow{f^{\mathcal{N}}} & N_s
 \end{array}$$

The top square and the square with the curved arrows commute because μ and ν are morphisms in $\Sigma\text{-Str}(\mathcal{C})$, providing the following equalities:

$$\begin{aligned}
 \theta(f^{\prod_F \mathcal{M}}(\mu_I(x))) &= \theta(\mu_I(f^{\prod_I \mathcal{M}_i}(x))) \\
 &= \nu_I(f^{\prod_I \mathcal{M}_i}(x)) \\
 &= f^{\mathcal{N}}(\nu_I(x)) \\
 &= f^{\mathcal{N}}(\theta(\mu_I(x))).
 \end{aligned}$$

From this, we derive that $\theta \circ f^{\prod_F \mathcal{M}} \circ \mu_I = f^{\mathcal{N}} \circ \theta \circ \mu_I$. Lemma 4.2 ensures that μ_I is an epimorphism, meaning that $\theta \circ f^{\prod_F \mathcal{M}} = f^{\mathcal{N}} \circ \theta$.

Second, consider a relation name $r : s_1 \times \dots \times s_n$ in R . The relation name r induces two subobjects $r^{\prod_I \mathcal{M}_i} : O_I \rightarrow (\prod_I \mathcal{M}_i)_{s_1} \times \dots \times (\prod_I \mathcal{M}_i)_{s_n}$ and $r^{\mathcal{N}} : O_N \rightarrow N_{s_1} \times \dots \times N_{s_n}$. Since ν is a morphism, we describe O_N using the internal logic of \mathcal{C} :

$$O_N = \{ \theta(\mu_I(x_I)) \mid x_I \in R^{\prod_I \mathcal{M}_i} \}.$$

Thus, the following statement holds in the internal language of \mathcal{C} :

$$\forall x_I \in \prod_I \mathcal{M}_i, \mu_I(x_i) \in r^{\prod_F \mathcal{M}} \Rightarrow \theta(\mu_I(x_I)) \in r^{\mathcal{N}}.$$

which proves that there exists a morphism $O_F \rightarrow O_N$ such that the diagram

$$\begin{array}{ccc}
 O_F & \xrightarrow{r^{\prod_F \mathcal{M}}} & \prod_F M_{s_1} \times \dots \times \prod_F M_{s_n} \\
 \downarrow & & \downarrow (\theta_{s_1}, \dots, \theta_{s_n}) \\
 O_N & \xrightarrow{r^{\mathcal{N}}} & N_{s_1} \times \dots \times N_{s_n}
 \end{array}$$

commutes. □

The second proposition relates filtered products of subobjects and subobjects of filtered products for atomic formulae.

Proposition 4.3. *Let $r(\vec{t})$ be a Σ -atomic formula and \vec{x} a suitable context for it. Then, $\{\vec{x} \mid r(\vec{t})(\vec{x})\}_{\prod_F M}$ is the filtered product of $(\{\vec{x} \mid r(\vec{t})(\vec{x})\}_{M_i})_{i \in I}$ in \mathcal{C} .*

Proof. This is a direct application that filtered colimits commute with finite limits in cocomplete categories [13, Theorem 1 (p. 215), Sect. 2, Chap. IX]. \square

4.2 Fundamental Theorem

In Set, Loś's theorem (see Def. 1.1 in the introduction) associates each I -sequence $(a_i)_{i \in I}$ with an index J of the filter. Therefore, an index for a set of I -sequences corresponds to the intersection of all associated indices. This correspondence is only valid when the set of associated indices remains finite, as filters are only stable by finite intersections. To address this finiteness constraint in our categorical context, we consider generalized elements of the form $\alpha : U \rightarrow (\prod_I M_i)_{\vec{x}}$, such that U is both finitely presentable and Noetherian.

Theorem 4.4 (Loś's theorem). *Let \mathcal{C} be a locally finitely presentable elementary topos with filtered products such that all finitely presentable objects are Noetherian. Let I be a set and $(M_i)_{i \in I}$ be a family of Σ -structures in \mathcal{C} . Let F be an ultrafilter on I such that for every $J \in F$, the canonical projection $p_{I,J} : \prod_I M_i \rightarrow \prod_J M_j$ is an epimorphism. Let φ be a Σ -formula with a suitable context \vec{x} and $\alpha : U \rightarrow (\prod_I M_i)_{\vec{x}}$ be a monomorphism such that U is a finitely presentable object. Then,*

$$\prod_F M \models_{(\mu_I)_{\vec{x}} \circ \alpha} \vec{x}.\varphi \text{ iff } \{i \in I \mid M_i \models_{(p_{I,i})_{\vec{x}} \circ \alpha} \vec{x}.\varphi\} \in F$$

The proof of Theorem 4.4 rests on the following lemmas.

Lemma 4.5. *Assuming the context of Def. 4.4, any morphism $g : A \rightarrow B$, such that $A \neq \emptyset$, can be factorized through any epimorphism $f : X \rightarrow B$.*

Proof. We recall that in a topos \mathcal{C} , the epimorphism $f : X \rightarrow B$ is regular, meaning it is the coequalizer of some parallel pair of morphisms. In particular, a regular epimorphism satisfies $\forall b : B, \exists x : X, f(x) = b$ in a topos. Consider a morphism $g : A \rightarrow B$ and a subobject S of the pullback $A \times_B X$ satisfying the two following properties, expressed using the internal language of \mathcal{C} :

$$\begin{aligned} & \forall a : A, \exists x : X, (a, x) \in_{A \times X} S, \\ & \forall a : A, \forall x : X, \forall x' : X, (a, x) \in_{A \times X} S \wedge (a, x') \in_{A \times X} S \implies a = a'. \end{aligned}$$

As f is an epimorphism, such a S exists. By construction of the pullback, for all $(a, x) \in S$, $g(a) = f(x)$. By definition of S , for all $a : A$, there is a unique $x : X$ such that $g(a) = f(x)$. Thus, we can consider the morphism $h : A \rightarrow X$ such that for all $(a, x) : A \times X$, if $(a, x) \in_{A \times X} S$, then $h(a) = x$. \square

As in Set, several S satisfying the given property might exist, meaning that h is intrinsically not unique, but its existence suffices for our needs.

Lemma 4.6. *Assuming the context of Def. 4.4, for any $J \subseteq I$,*

$$\{\tilde{x} \mid \varphi(\tilde{x})\}_{\prod_J M_j} = \prod_J \{\tilde{x} \mid \varphi(\tilde{x})\}_{M_j}.$$

Proof. By inversion of product and pullback. \square

We can now prove Def. 4.4.

Proof. The proof is done by structural induction on φ .

- φ is of the form $r(\vec{t})$.

(\Rightarrow) We suppose that $\prod_F \mathcal{M} \vDash_{(\mu_I)_{\tilde{x}} \circ \alpha} \tilde{x}.r(\vec{t})$, meaning that there exists a morphism $m : U \rightarrow \{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{\prod_F M}$ such that $[[\tilde{x}.r(\vec{t})]]_{\prod_F \mathcal{M}} \circ m = (\mu_I)_{\tilde{x}} \circ \alpha$. By proposition 4.3, it follows that $\{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{\prod_F M}$ is the filtered product of $(\{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{M_i})_{i \in I}$ modulo F in \mathcal{C} . Let ν be the colimit associated with the filtered product of $(\{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{M_i})_{i \in I}$. Since U is a finitely presentable object (Definition 2.4), the morphism $m : U \rightarrow \{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{\prod_F M}$ factors (essentially uniquely) through some morphism $\nu_J : \prod_J \{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{M_j} \rightarrow \{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{\prod_F M}$. By Lemma 4.6, ν_J is a morphism $\{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{\prod_J M_j} \rightarrow \{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{\prod_F M}$. Thus, there exists $J \in F$ and a morphism $\delta : U \rightarrow \{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{\prod_J M_j}$ such that the following diagram commutes:

$$\begin{array}{ccccccc} & & & & m & & \\ & & & & \curvearrowright & & \\ & & & & \{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{\prod_J M_j} & \xrightarrow{\nu_J} & \{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{\prod_F M} & \longrightarrow & \mathbf{1} \\ & & & & \downarrow [[\tilde{x}.r(\vec{t})]]_{\prod_J \mathcal{M}_j} & & \downarrow [[\tilde{x}.r(\vec{t})]]_{\prod_F \mathcal{M}} & & \downarrow \text{true} \\ U & \xrightarrow{\alpha} & (\prod_I M_i)_{\tilde{x}} & \xrightarrow{(p_{I,j})_{\tilde{x}}} & (\prod_J M_j)_{\tilde{x}} & \xrightarrow{(\mu_J)_{\tilde{x}}} & (\prod_F M)_{\tilde{x}} & \longrightarrow & \Omega. \end{array}$$

Thus, $\prod_J M_j \vDash_{(p_{I,j})_{\tilde{x}} \circ \alpha} \tilde{x}.r(\vec{t})$, meaning that for all j in J , $M_j \vDash_{(p_{I,j})_{\tilde{x}} \circ \alpha} \tilde{x}.r(\vec{t})$, i.e., $J \subseteq \{i \in I \mid M_i \vDash_{(p_{I,i})_{\tilde{x}} \circ \alpha} \tilde{x}.r(\vec{t})\}$. Since F is a filter, we can conclude that $\{i \in I \mid M_i \vDash_{(p_{I,i})_{\tilde{x}} \circ \alpha} \tilde{x}.r(\vec{t})\} \in F$.

(\Leftarrow) By Proposition 4.3, for every $J \in F$, the following diagram

$$\begin{array}{ccc} \{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{\prod_J M_j} & \xrightarrow{\nu_J} & \{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{\prod_F M} \\ \downarrow [[\tilde{x}.r(\vec{t})]]_{\prod_J \mathcal{M}_j} & & \downarrow [[\tilde{x}.r(\vec{t})]]_{\prod_F \mathcal{M}} \\ (\prod_J M_j)_{\tilde{x}} & \xrightarrow{(\mu_J)_{\tilde{x}}} & (\prod_F M)_{\tilde{x}} \end{array}$$

commutes, where ν is the colimit associated with the filtered product of $(\{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{M_i})_{i \in I}$. Thus, it follows that $\text{Im}((\mu_I)_{\tilde{x}} \circ \alpha) \leq \{\tilde{x} \mid r(\vec{t})(\tilde{x})\}_{\prod_F M}$.

- φ is of the form $\psi \wedge \chi$.

(\Rightarrow) By Def. 3.6, $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha}} \vec{x}.\psi \wedge \chi$ implies that $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha}} \vec{x}.\psi$ and $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha}} \vec{x}.\chi$. By the induction hypothesis, it follows that $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha}} \vec{x}.\psi\}$ and $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha}} \vec{x}.\chi\}$ are in F . Since filters are closed under intersection, $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha}} \vec{x}.\psi \wedge \chi\}$ is also in F .

(\Leftarrow) Since $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha}} \vec{x}.\psi \wedge \chi\} \subseteq \{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha}} \vec{x}.\psi\}$, F is a filter, and $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha}} \vec{x}.\psi \wedge \chi\}$ is in F , it follows that the set $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha}} \vec{x}.\psi\}$ is in F . Similarly, $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha}} \vec{x}.\chi\}$ is also in F . By the induction hypothesis, it holds that $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha}} \vec{x}.\psi$ and $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha}} \vec{x}.\chi$. According to Def. 3.6, we conclude that $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha}} \vec{x}.\psi \wedge \chi$.

- φ is of the form $\psi \vee \chi$.

(\Rightarrow) By Def. 3.6, there are two morphisms $p:V \rightarrow U$ and $q:W \rightarrow U$ such that $p+q$ is an epimorphism, $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha\circ p}} \vec{x}.\psi$, and $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha\circ q}} \vec{x}.\chi$. By the induction hypothesis, it follows that $J_\psi = \{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha\circ p}} \vec{x}.\psi\}$ and $J_\chi = \{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha\circ q}} \vec{x}.\chi\}$ are in F . Thus $J_\psi \cup J_\chi$ is in F , i.e., $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha}} \vec{x}.\psi \vee \chi\}$ is in F .

(\Leftarrow) Let us suppose that the set $J = \{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha}} \vec{x}.\psi \vee \chi\}$ is in F . Then, for all $j \in J$, $\mathcal{M}_j \models_{(p_{I,j})_{\vec{x}\circ\alpha}} \vec{x}.\psi \vee \chi$. By Lemma 4.6, it holds that $\prod_J \mathcal{M}_j \models_{(p_{I,J})_{\vec{x}\circ\alpha}} \vec{x}.\psi \vee \chi$. By Def. 3.6, there are morphisms $p:V \rightarrow U$ and $q:W \rightarrow U$ such that $p+q:V+W \rightarrow U$ is an epimorphism, and both $\prod_J \mathcal{M}_j \models_{(p_{I,J})_{\vec{x}\circ\alpha\circ p}} \vec{x}.\psi$ and $\prod_J \mathcal{M}_j \models_{(p_{I,J})_{\vec{x}\circ\alpha\circ q}} \vec{x}.\chi$. By definition of products, for any $j \in J$ it holds that $\mathcal{M}_j \models_{(p_{I,j})_{\vec{x}\circ\alpha\circ p}} \vec{x}.\psi$ and $\mathcal{M}_j \models_{(p_{I,j})_{\vec{x}\circ\alpha\circ q}} \vec{x}.\chi$. In particular, it follows that

$$\begin{aligned} - J &\subseteq J_\psi = \{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha\circ p}} \vec{x}.\psi\} \text{ and} \\ - J &\subseteq J_\chi = \{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\vec{x}\circ\alpha\circ q}} \vec{x}.\chi\}. \end{aligned}$$

Since F is an ultrafilter, we deduce that the sets J_ψ and J_χ are in F . By the induction hypothesis, it holds that $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha\circ p}} \vec{x}.\psi$ and $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha\circ q}} \vec{x}.\chi$, $p:V \rightarrow U$ and $q:W \rightarrow U$ such that $p+q:V+W \rightarrow U$ is an epimorphism. By Def. 3.6, we can conclude that $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha}} \vec{x}.\psi \vee \chi$.

- φ is of the form $\psi \Rightarrow \chi$.

(\Rightarrow) Let us suppose that $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha}} \vec{x}.\psi \Rightarrow \chi$. By Def. 3.6, it follows that for every morphism $p:V \rightarrow U$, if $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha\circ p}} \vec{x}.\psi$, then $\prod_F \mathcal{M} \models_{(\mu_I)_{\vec{x}\circ\alpha\circ p}} \vec{x}.\chi$. By the same argument as in Corollary 3.10, we can restrict to monomorphisms $p:V \rightarrow U$. In particular, if we consider the two following sets:

- $\Gamma = \{V \in \text{Sub}(U) \mid \prod_F \mathcal{M} \not\models_{(\mu_I)_{\bar{x}} \circ \alpha \circ (V \rightarrow U)} \bar{x}.\psi\}$ and
- $\Delta = \{V \in \text{Sub}(U) \mid \prod_F \mathcal{M} \models_{(\mu_I)_{\bar{x}} \circ \alpha \circ (V \rightarrow U)} \bar{x}.\chi\}$,

then $\prod_F \mathcal{M} \models_{(\mu_I)_{\bar{x}} \circ \alpha} \bar{x}.\psi \Rightarrow \chi$ means that $\text{Sub}(U) = \Gamma \cup \Delta$. For V in Γ , the induction hypothesis implies that $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}} \circ \alpha \circ (V \rightarrow U)} \bar{x}.\psi\}$ is not in F . Since F is an ultrafilter, $J_V = \{i \in I \mid \mathcal{M}_i \not\models_{(p_{I,i})_{\bar{x}} \circ \alpha \circ (V \rightarrow U)} \bar{x}.\psi\}$ is in F . As U is Noetherian, Proposition 2.7 ensures that Γ is finite, and then $J_\Gamma = \bigcap_{V \in \Gamma} J_V$ is also in F . With similar arguments, it follows that $J_\Delta = \{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}} \circ \alpha \circ (V_\Delta \rightarrow U)} \bar{x}.\chi\}$ is in F , where V_Δ is the union of all subobjects in Δ . By the monotonicity property (see Proposition 3.4), for every $j \in J_\Delta$ and every $V \in \Delta$ it holds that $\mathcal{M}_j \models_{(p_{I,j})_{\bar{x}} \circ \alpha \circ (V \rightarrow U)} \bar{x}.\chi$. Let us consider $J = J_\Gamma \cap J_\Delta$, $j \in J$ and $V \in \text{Sub}(U)$. If V is in Γ , then $\mathcal{M}_j \not\models_{(p_{I,j})_{\bar{x}} \circ \alpha \circ (V \rightarrow U)} \bar{x}.\psi$. Otherwise, V is in Δ , meaning that $\mathcal{M}_j \models_{(p_{I,j})_{\bar{x}} \circ \alpha \circ (V \rightarrow U)} \bar{x}.\psi$ and $\mathcal{M}_j \not\models_{(p_{I,j})_{\bar{x}} \circ \alpha \circ (V \rightarrow U)} \bar{x}.\chi$. By Def. 3.6, we can then deduce that for every $j \in J$, $\mathcal{M}_j \models_{(p_{I,j})_{\bar{x}} \circ \alpha} \bar{x}.\psi \Rightarrow \chi$. Since J is in F and F is a filter, we can conclude that $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}} \circ \alpha} \bar{x}.\psi \Rightarrow \chi\} \in F$.

(\Leftarrow) Suppose that $J = \{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}} \circ \alpha} \bar{x}.\psi \Rightarrow \chi\}$ is in F . Let $p: V \rightarrow U$ be a morphism such that $\prod_F \mathcal{M} \models_{(\mu_I)_{\bar{x}} \circ \alpha \circ p} \bar{x}.\psi$. By induction hypothesis, this means that the set $J_\psi = \{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}} \circ \alpha \circ p} \bar{x}.\psi\}$ is in F . Since F is a filter, it follows that $L = J \cap J_\psi$ is in F . In particular, for $j \in L$, it holds that $\mathcal{M}_j \models_{(p_{I,j})_{\bar{x}} \circ \alpha} \bar{x}.\psi \Rightarrow \chi$ and $\mathcal{M}_j \models_{(p_{I,j})_{\bar{x}} \circ \alpha \circ p} \bar{x}.\psi$. By Def. 3.6, it follows that $\mathcal{M}_j \models_{(p_{I,j})_{\bar{x}} \circ \alpha \circ p} \bar{x}.\chi$. Thus, L is a subset of $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}} \circ \alpha \circ p} \bar{x}.\chi\}$. Since F is a filter, it holds that $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}} \circ \alpha \circ p} \bar{x}.\chi\}$ is in F . By induction hypothesis, it follows that $\prod_F \mathcal{M} \models_{(\mu_I)_{\bar{x}} \circ \alpha \circ p} \bar{x}.\chi$. By Def. 3.6, we can conclude that $\prod_F \mathcal{M} \models_{(\mu_I)_{\bar{x}} \circ \alpha} \bar{x}.\psi \Rightarrow \chi$.

- φ is of the form $\neg\psi$. The result follows from the fact that $\neg\psi \equiv \psi \Rightarrow \perp$ and that the only generalized element for which any Σ -structure \mathcal{M} satisfied $\bar{x}.\perp$ is $\emptyset \rightarrow M_{\bar{x}}$.
- φ is of the form $\exists y.\psi$.

(\Rightarrow) Suppose that $\prod_F \mathcal{M} \models_{(\mu_I)_{\bar{x}} \circ \alpha} \bar{x}.\exists y.\psi$. By Def. 3.6, there exists an epimorphism $p: V \rightarrow U$ and a generalized element $\beta: V \rightarrow (\prod_F M)_s$ such that $\prod_F \mathcal{M} \models_{((\mu_I)_{\bar{x}} \circ \alpha \circ p, \beta)} \bar{x}y.\psi$. From Lemma 4.5, β factorizes through $(\mu_I)_s$, i.e., there exists $\delta: V \rightarrow (\prod_I \mathcal{M}_i)_s$ such that $\beta = (\mu_I)_s \circ \delta$. Then, $\prod_F \mathcal{M} \models_{(\mu_I)_{\bar{x}y} \circ (\alpha \circ p, \delta)} \bar{x}y.\psi$. By the induction hypothesis, $J = \{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}y} \circ (\alpha \circ p, \delta)} \bar{x}y.\psi\}$ is in F . Consider j in J . Then, we have that $\mathcal{M}_j \models_{(p_{I,j})_{\bar{x}y} \circ (\alpha \circ p, \delta)} \bar{x}y.\psi$, with $p: V \rightarrow U$ epimorphism and $(p_{I,j})_s \circ \delta: V \rightarrow (M_j)_s$ generalized element. By Def. 3.6, it follows that $\mathcal{M}_j \models_{(p_{I,j})_{\bar{x}} \circ \alpha} \bar{x}.\exists y.\psi$. Thus, it holds that $J \subseteq \{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}} \circ \alpha} \bar{x}.\exists y.\psi\}$. Since F is a filter, we can conclude that $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}} \circ \alpha} \bar{x}.\exists y.\psi\} \in F$.

(\Leftarrow) Suppose that $J = \{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}} \circ \alpha} \bar{x}.(\exists y.\psi)\} \in F$. Then, for all $j \in J$, $\mathcal{M}_j \models_{(p_{I,j})_{\bar{x}} \circ \alpha} \bar{x}.(\exists y.\psi)$. By Lemma 4.6, it holds that $\prod_J \mathcal{M}_j \models_{(p_{I,J})_{\bar{x}} \circ \alpha} \bar{x}.(\exists y.\psi)$. By Def. 3.6, there exists an epimorphism $p: V \rightarrow U$ and a generalized element $\beta: V \rightarrow (\prod_J M_j)_s$ such that $\prod_J \mathcal{M}_j \models_{((p_{I,J})_{\bar{x}} \circ \alpha \circ p, \beta)} \bar{x}y.\psi$. From Lemma 4.5, β factorizes through $(p_{I,J})_s$, i.e., there exists $\delta: V \rightarrow (\prod_I \mathcal{M}_i)_s$ such that $\beta = (p_{I,J})_s \circ \delta$. Thus, it holds that $\prod_J \mathcal{M}_j \models_{(p_{I,J})_{\bar{x}y} \circ (\alpha \circ p, \delta)} \bar{x}y.\psi$. By definition of products, it follows that for every $j \in J$, $\mathcal{M}_j \models_{(p_{I,j})_{\bar{x}y} \circ (\alpha \circ p, \delta)} \bar{x}y.\psi$. In particular, J is a subset of $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}y} \circ (\alpha \circ p, \delta)} \bar{x}y.\psi\}$. Since F is a filter, it holds that $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}y} \circ (\alpha \circ p, \delta)} \bar{x}y.\psi\} \in F$. By the induction hypothesis, we obtain $\prod_F \mathcal{M} \models_{(\mu_I)_{\bar{x}y} \circ (\alpha \circ p, \delta)} \bar{x}y.\psi$, with p epimorphism $V \rightarrow U$ and $(\mu_I)_s \circ \delta$ generalized element $V \rightarrow (\prod_F M)_s$. By Def. 3.6, we can conclude that $\prod_F \mathcal{M} \models_{(\mu_I)_{\bar{x}} \circ \alpha} \bar{x}.(\exists y.\psi)$.

- φ is of the form $\forall y.\psi$.

(\Rightarrow) By contraposition we suppose that the set $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}} \circ \alpha} \bar{x}.(\forall y.\psi)\}$ is not in F . As F is an ultrafilter, $J = \{i \in I \mid \mathcal{M}_i \not\models_{(p_{I,i})_{\bar{x}} \circ \alpha} \bar{x}.(\forall y.\psi)\}$ is in F . Then, for all $j \in J$, $\mathcal{M}_j \not\models_{(p_{I,j})_{\bar{x}} \circ \alpha} \bar{x}.(\forall y.\psi)$. By Def. 3.6, we have for every $j \in J$ that there exists a morphism $p_j: V_j \rightarrow U$ and a generalized element $\beta_j: V_j \rightarrow (M_j)_s$ such that $\mathcal{M}_j \not\models_{((p_{I,j})_{\bar{x}} \circ \alpha \circ p_j, \beta_j)} \bar{x}y.\psi$. From Lemma 4.5, β_j factorizes through $(p_{I,j})_y$, i.e., there exists a morphism $\delta_j: V_j \rightarrow (\prod_I \mathcal{M}_i)_y$ such that $\beta_j = (p_{I,j})_y \circ \delta_j$. By factorization, it holds that $\mathcal{M}_j \not\models_{(p_{I,j})_{\bar{x}y} \circ (\alpha \circ p_j, \delta_j)} \bar{x}y.\psi$. Let V be the colimit of $\{V_j \mid j \in J\}$ and $i_j: V_j \rightarrow V$ the canonical injections. Then, by the universal property of colimit, there is a unique morphism $q: V \rightarrow U$ such that $p_j = q \circ i_j$. Likewise, there is a unique morphism $\delta: V \rightarrow (\prod_I \mathcal{M}_i)_y$ such that $\delta_j = \delta \circ i_j$. Hence, we have that $\mathcal{M}_j \not\models_{(p_{I,j})_{\bar{x}y} \circ (\alpha \circ q \circ i_j, \delta \circ i_j)} \bar{x}y.\psi$, and then $\mathcal{M}_j \not\models_{(p_{I,j})_{\bar{x}y} \circ (\alpha \circ q, \delta)} \bar{x}y.\psi$. By the induction hypothesis, we then deduce that $\prod_F \mathcal{M} \not\models_{(\mu_I)_{\bar{x}y} \circ (\alpha \circ q, \delta)} \bar{x}y.\psi$, from which we conclude that $\prod_F \mathcal{M} \not\models_{(\mu_I)_{\bar{x}} \circ \alpha} \bar{x}.(\forall y.\psi)$.

(\Leftarrow) By contraposition, we suppose that $\prod_F \mathcal{M} \not\models_{(\mu_I)_{\bar{x}} \circ \alpha} \bar{x}.(\forall y.\psi)$. By Def. 3.6, there exists a morphism $p: V \rightarrow U$ and a generalized element $\beta: V \rightarrow (\prod_F M)_s$ such that $\prod_F \mathcal{M} \not\models_{((\mu_I)_{\bar{x}} \circ \alpha \circ p, \beta)} \bar{x}y.\psi$. From Lemma 4.5, β factorizes through $(\mu_I)_y$, i.e., there exists a morphism $\delta: V \rightarrow (\prod_I \mathcal{M}_i)_s$ such that $\beta = (\mu_I)_y \circ \delta$. By factorization and product, it holds that $\prod_F \mathcal{M} \not\models_{(\mu_I)_{\bar{x}y} \circ (\alpha \circ p, \delta)} \bar{x}y.\psi$. By the induction hypothesis, it follows that the set $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}y} \circ (\alpha \circ p, \delta)} \bar{x}y.\psi\}$ is not in F . Since F is an ultrafilter, the set $J = \{i \in I \mid \mathcal{M}_i \not\models_{(p_{I,i})_{\bar{x}y} \circ (\alpha \circ p, \delta)} \bar{x}y.\psi\}$ is in F . Thus, for any $j \in J$, it holds that $\mathcal{M}_j \not\models_{(p_{I,j})_{\bar{x}y} \circ (\alpha \circ p, \delta)} \bar{x}y.\psi$, with p morphism $V \rightarrow U$ and $(p_{I,j})_s \circ \delta$ generalized element $V \rightarrow (M_j)_s$. By Def. 3.6, we obtain that $\mathcal{M}_j \not\models_{(p_{I,j})_{\bar{x}} \circ \alpha} \bar{x}.(\forall y.\psi)$. In particular, $J \subseteq \{i \in I \mid \mathcal{M}_i \not\models_{(p_{I,i})_{\bar{x}} \circ \alpha} \bar{x}.(\forall y.\psi)\}$. As F is an ultrafilter, we deduce that the set $\{i \in I \mid \mathcal{M}_i \not\models_{(p_{I,i})_{\bar{x}} \circ \alpha} \bar{x}.(\forall y.\psi)\}$

is in F and conclude that the set $\{i \in I \mid \mathcal{M}_i \models_{(p_{I,i})_{\bar{x}} \circ \alpha} \bar{x}.(\forall y.\psi)\}$ is not in F .

□

Upon reading the proof of Def. 4.4, the reader will realize that the condition imposed on the domain U of the generalized element to be Noetherian is no longer necessary when restricting the study to

- Cartesian formulae (closed under finite conjunction),
- regular formulae (closed under finite conjunction and existential quantifier),
- coherent formulae (closed under finite conjunction, finite disjunction, and existential quantifier).

Def. 4.4 generalizes Loś's theorem to Kripke-Joyal semantics for the internal language of topoi. It implies its set-theoretic version, i.e., Def. 1.1. Indeed \mathbf{Set} is a locally finitely presentable elementary topos with filtered products in which the canonical projections from any products are epimorphism. Besides, an I -sequence $(a_i)_{i \in I}$ in \mathbf{Set} corresponds to a generalized element $\alpha : \mathbf{1} \rightarrow (\prod_I M_i)_{\bar{x}}$. Applying Def. 4.4 to such generalized elements α yields Def. 1.1.

4.3 Compactness Theorem

The compactness theorem deals with a set of sentences, which, in FOL, corresponds to formulae without free variables. In other words, a FOL formula is a sentence if $[\]$ is a suitable context for it, meaning that its interpretation in a Σ -structure \mathcal{M} is a morphism with codomain $M_{[\]}$, that is the terminal object $\mathbf{1}$ of \mathcal{C} . Given a sentence φ , the generalized elements of interests for φ are $\alpha : U \rightarrow (\prod_I M_i)_{[\]}$, i.e., $\alpha : U \rightarrow \mathbf{1}$.

Definition 4.7 (Sentence). *A formula φ is a **sentence** if $[\]$ is a suitable context for φ .*

When φ is a sentence, we write $\mathcal{M} \models \varphi$ instead of $\mathcal{M} \models [\].\varphi$.

Proposition 4.8. *If φ is a sentence, then:*

$$\prod_F \mathcal{M} \models \varphi \text{ iff } \{i \in I \mid \mathcal{M}_i \models \varphi\} \in F.$$

Proof. The proof is a direct application of Def. 4.4, exploiting the fact that the diagrams

$$\begin{array}{ccccccc}
 & & \{\bar{x} \mid \varphi(\bar{x})\}_{\prod_I M_i} & \longrightarrow & \{\bar{x} \mid \varphi(\bar{x})\}_{\prod_F M} & \longrightarrow & \mathbf{1} \\
 & \nearrow & \downarrow \llbracket \bar{x}.\varphi \rrbracket_{\prod_I \mathcal{M}_i} & & \downarrow \llbracket \bar{x}.\varphi \rrbracket_{\prod_F \mathcal{M}} & & \downarrow \text{true} \\
 U & \xrightarrow{\alpha} & (\prod_I M_i)_{\bar{x}} & \xrightarrow{(\mu_I)_{\bar{x}}} & (\prod_F M)_{\bar{x}} & \longrightarrow & \Omega
 \end{array}$$

and

$$\begin{array}{ccccccc}
& & \{\bar{x} \mid \varphi(\bar{x})\}_{\prod_I M_i} & \longrightarrow & \{\bar{x} \mid \varphi(\bar{x})\}_{M_i} & \longrightarrow & \mathbf{1} \\
& \nearrow & \downarrow \llbracket \bar{x}.\varphi \rrbracket_{\prod_I \mathcal{M}_i} & & \downarrow \llbracket \bar{x}.\varphi \rrbracket_{\mathcal{M}_i} & & \downarrow \text{true} \\
U & \xrightarrow{\alpha} & (\prod_I M_i)_{\bar{x}} & \xrightarrow{(p_{I,i})_{\bar{x}}} & (M_i)_{\bar{x}} & \longrightarrow & \Omega
\end{array}$$

both become

$$\begin{array}{ccccccc}
& & \mathbf{1} & \xrightarrow{\text{id}_1} & \mathbf{1} & \xrightarrow{\text{id}_1} & \mathbf{1} \\
& \nearrow & \downarrow \text{id}_1 & & \downarrow \text{id}_1 & & \downarrow \text{true} \\
U & \xrightarrow{\alpha} & \mathbf{1} & \xrightarrow{\text{id}_1} & \mathbf{1} & \xrightarrow{\text{true}} & \Omega
\end{array}$$

when φ is a sentence. □

We can deduce the compactness theorem from Proposition 4.8

Theorem 4.9 (Compactness). *A set of sentences T has a model if and only if every finite subset of T has a model.*

Proof. The proof is a pastiche of its set-theoretic variant as a corollary from Loś's theorem. The implication is obvious: a model of T is a model of all subsets of T , in particular the finite ones.

For the converse, we suppose that T is infinite and that every finite subset i of T admits a model \mathcal{M}_i and write I for the set of all finite subsets of T . For $i \in I$, we write i^* for superset closure of i , $\{j \in I \mid i \subseteq j\}$. Let F be the subset of $\mathcal{P}(I)$ defined as $F = \{X \subseteq I \mid \exists i \in I, i^* \subseteq X\}$. Then F is a filter of I different from $\mathcal{P}(I)$, leveraging that $i^* \cap j^* = (i \cup j)^*$ for the closure under finite intersections. By Zorn's lemma, F can be extended into an ultrafilter U over I . Let φ be a sentence in T , then $\{\varphi\}$ is an element of I . Thus, for all $i \in \{\varphi\}^*$, \mathcal{M}_i is a model for φ . Thus $\{\varphi\}^* \subseteq \{i \in I \mid \mathcal{M}_i \models \varphi\}$, i.e., $\{i \in I \mid \mathcal{M}_i \models \varphi\} \in F$, meaning that $\{i \in I \mid \mathcal{M}_i \models \varphi\} \in U$. From Proposition 4.8, it follows that $\prod_U \mathcal{M} \models \varphi$. Thereafter, $\prod_U \mathcal{M} \models T$, i.e., T has a model. □

5 Conclusion

In this paper, we explored ultraproducts of first-order logic (FOL) structures categorically, extending Loś's theorem from set-theoretic models to Kripke-Joyal semantics for the internal logic of topoi. Although topoi allow for reasoning akin to sets, their inherent abstract nature required some additional conditions for the theorem to hold: (1) the topos must be locally finitely presentable as filters are only closed under finite intersections ; (2) the topos must have filtered products to ensure the existence of ultraproducts ; (3) the canonical projections between model products should be epimorphisms to use the semantic rules for the disjunctions and existential quantifiers ; (4) the generators must have finitely many subobjects, which is needed for proving the case of the implication. We showed that our extension of Loś's theorem naturally implies that in Set. As an immediate application, we demonstrated how this extension yields the compactness theorem for Kripke-Joyal semantics in topoi.

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